Stability of Deeba Functional Equations on Amenable Semigroups: A Forti Method Approach

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Abstract:
In this article, we study the stability of Deeba functional equation on amenable groups. To this end, we have used Forti method. The main results of this paper generalize the works of Forti [9].

Keywords: Functional equations, stability, Forti method, Deeba functional equation, amenable group.

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1. Introduction
A classical question in the theory of functional equations is that "when is it true that a function which approximately satisfies a functional equation $\varepsilon$ must be somehow close to an exact solution of $\varepsilon$ ". Such a problem was formulated by Ulam [19] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [5]. It gave rise to the stability theory for functional equations. The result of Hyers was generalized by Aoki [1] for approximate additive functions and by Th.M. Rassias [6] for approximate linear functions. In 1983, Skof [18] considered the Hyers-Ulam stability of the quadratic functional equation:
\[
\| f(xy) + f(xy^{-1}) \| = 2f(x) + 2f(y) \tag{1.1}
\]

Where \( f \) maps a group \( G \) to an abelian group \( H \). As usual, each solution of equation (1.1) is called a quadratic function. But Skof restricted herself to studying the case where \( f \) maps a normed space to a Banach space.

In [2] Cholewa noticed that the theorem of Skofs is true if the relevant domain in replaced by an abelian group. The results of Skof and Cholewa were further generalized by Czerwik [3]. Further works on stability of the quadratic functional equation can be found in Fenyo [8], Czerwik [4], Czerwik and Dlutek [5], [6], Ger [10], Jung [13], Jung and Sahoo [14], and Rassias [17].

Let \( G \) be a group and \( X \) and \( Y \) be any two arbitrary Banach spaces over reals. Faiziev and Sahoo [7] proved that the quadratic functional equation is stable for the pair \((G,X)\) if and only if it is stable for the pair \((G,Y)\). In view of this result it is not important which Banach space is used on the range. Thus one may consider the stability of the quadratic functional equation on the pair \((G,R)\). Faiziev and Sahoo [7] proved that quadratic functional equation is not stable on the pair \((G,R)\) when \( G \) is any arbitrary group. It is well known (see Skof [18] and Cholewa [2]) that the quadratic functional equation is stable on the pair \((G,R)\) when \( G \) is an abelian group. Thus it is interesting to know on which noncommutative groups the quadratic functional equation is stable in the sense of Hyers-Ulam.

Faiziev and Sahoo [7] proved that quadratic functional equation is stable on n-abelian groups and \( T(2,k) \), where \( K \) is a commutative field. Further they also proved that every group can be embedded into a group in which the quadratic functional equation is stable. Yang [20] proved the
stability of quadratic functional equation on amenable groups. In an American Mathematical Society meeting, E. Y. Deeba of the University of Houston asked to find the general solution of the functional equation

$$f(x+y+z)+f(x)+f(y)+f(z)=f(x+y)+f(x+z)+f(y+z). \quad (1.2)$$

This functional equation is a variation of the quadratic functional equation. Kannappan [15] showed that the general solution $f : V \to K$ of the above functional equation is of the form

$$f(x) = B(x,x) + A(x),$$

where $B : V \times V \to K$ is a symmetric biadditive function and $A : V \to K$ is an additive function, $V$ is a vector space, and $K$ is a field of characteristic different from two (or of characteristic zero). The Hyers- Ulam stability of the equation (1.2) was investigated by Jung [12]. The multiplicative form of Deeba quadratic functional equation(1.2) on an arbitrary group $G$ or on a semigroup $S$ is

$$f(xyz)+f(x)+f(y)+f(z)=f(xy)+f(xz)+f(yz). \quad (1.3)$$

The stability of this system of equation was investigated by Faiziev and Sahoo [8]. They considered the stability of the functional equation (1.3) for the pair $(S,E)$ when $S$ is an arbitrary semigroup and $E$ is a real Banach space. If $X$ is another real Banach space, then they proved that the functional equation (1.3) is stable for the pair $(S,X)$ if and only if it is stable for the pair $(S,E)$. 
They proved that, in general, the equation (1.3) is not stable on semigroups. However, this equation (1.3) is stable on periodic semigroups as well as abelian semigroups. They also showed that any semigroup with left (or right) cancellation law can be embedded into a semigroup with left (or right) cancellation law where the equation (1.3) is stable [8]. Therefore, a question naturally arises: is equation (1.3) stable on amenable semigroups? In this note, we will give an affirmative answer.

2. Main Results

Definition 2.1. Let \((G,\cdot)\) be a semigroup (or group) and \(B(G)\) denote the space of all bounded complex-valued functions on \(G\) with the norm 
\[
\|f\| = \sup \{f(x) | x \in G\}
\]
A linear functional \(m\) on \(B(G)\) is called a right invariant mean if the following condition are satisfied:

i. \(m(\bar{f}) = \overline{m(f)}\) for all \(f \in B(G)\),

ii. \(\inf \{f(x) | x \in G\} \leq m(f) \leq \sup \{f(x) | x \in G\}\) for all real-valued \(f \in B(G)\),

iii. \(m(f_x) = m(f)\) for all \(x \in G\) and \(f \in B(G)\), where \(f_x(t) = f(tx)\).

If (iii) in the above definition is replaced with \(m(f_x) = m(f)\), where \(f_x(t) = f(xt)\) then \(m\) is called a left invariant mean.

When a right (left) invariant mean exists on \(B(G)\), we call \(G\) right (left) amenable. It is known that if \(G\) is a semigroup with both right and left invariant means, then there exists a two-sided
invariant mean on $B(G)$ and in this case $G$ is called amenable. It is also known that if $G$ is a group, then either right or left amenability of $G$ implies that $G$ is amenable. We remark that the norm of the functional $m$ is one and thus in this paper we suppose that $f \in B(G)$ [4]. G. L. Forti [9] proved the following theorem.

**Lemma 2.2 (Forti).** Assume that $(G, \cdot)$ is a right (left) amenable semigroup. If a function

$$f: G \rightarrow \mathbb{C}$$

satisfies

$$\left| f(x \cdot y) - f(x) - f(y) \right| \leq \delta$$

for some $\delta \geq 0$ and for all $x, y \in G$, then there exists a homomorphism $H: G \rightarrow \mathbb{C}$ such that

$$\left| f(x) - H(x) \right| \leq \delta$$

for all $x \in G$ [9].

**Proof.** Let $m: B(G) \rightarrow \mathbb{C}$ be a right invariant mean. We use the notation $m_x$ to indicate that the mean is to be applied with respect to the variable $x$. Define the function $H: G \rightarrow \mathbb{C}$ by

$$H(y) = m_x \left\{ f(x \cdot y) - f(x) \right\}$$

Using the right invariance and the linearity of the functional $m$, we have

$$H(y) + H(z) = m_x \left\{ f(x \cdot y) - f(x) \right\} + m_x \left\{ f(x \cdot z) - f(x) \right\}$$

$$= m_x \left\{ f(x \cdot y \cdot z) - f(x \cdot z) + f(x \cdot z) - f(x) \right\}$$

$$= m_x \left\{ f(x \cdot y \cdot z) - f(x) \right\}$$

$$= H(y \cdot z),$$
so that $H$ is a homomorphism. We now get

\[
\left| f(y) - H(y) \right| = \left| f(y) - m_x \left\{ f(x \cdot y) - f(x) \right\} \right| \\
= \left\| m_x \right\| \left| f(x \cdot y) - f(x) - f(y) \right| \\
\leq \delta
\]

for all $y \in G$. The proof for the case of a left invariant mean is similar [9].

Now, in view of the last theorem we investigate the stability of Deeba functional equation on amenable groups.

**Theorem 2.3.** Assume that $G$ is a left amenable semigroup and assume that $f: G \rightarrow \mathbb{C}$ is a mapping satisfies

\[
\left| f(xyz) + f(x) + f(y) + f(z) - f(xy) - f(xz) - f(yz) \right| \leq \delta
\]

for $\delta > 0$ and for all $x, y \in G$, then there exists a multiplicative Deeba function $\varphi: G \rightarrow \mathbb{C}$ such that

\[
\left| \varphi(x) - f(x) \right| \leq \delta
\]

for all $x \in G$.

**Proof.** In view of the Lemma 2.2 we define the function $\varphi: G \rightarrow \mathbb{C}$ by

\[
\varphi(x) = m_y \left\{ f \cdot f \right\} + m_x \left\{ f \cdot f \right\} - m_z \left\{ f \cdot f \right\},
\]

then we have
\[ \phi(xpq) = m_y \left\{ xpq f - f \right\} + m_z \left\{ xpq f - f \right\} - m_{yz} \left\{ xpq f - f \right\}, \]

for all \( x, y, z, p, q \in G \).

We can write the first term of the last equation in the form:

\[ m_y \left\{ xpq f - f \right\} = m_y \left\{ -xf f + x f - xq f + x f - x f + f \right\} + x f f - f + xq f - f + xpq f - f \} \]

Using this method for two next term we have:

\[ \phi(xpq) = m_y \left\{ xpq f - f \right\} + m_z \left\{ xpq f - f \right\} + m_{yz} \left\{ xpq f - f \right\} \]

\[ = -m_y \left\{ x f - f \right\} - m_z \left\{ x f - f \right\} + m_{yz} \left\{ x f - f \right\} \]

\[ -m_y \left\{ xq f - f \right\} - m_z \left\{ xq f - f \right\} + m_{yz} \left\{ xq f - f \right\} \]

\[ -m_y \left\{ x f - f \right\} - m_z \left\{ x f - f \right\} + m_{yz} \left\{ x f - f \right\} \]

\[ + m_y \left\{ x f - f \right\} + m_z \left\{ x f - f \right\} - m_{yz} \left\{ x f - f \right\} \]

\[ + m_y \left\{ xq f - f \right\} + m_z \left\{ xq f - f \right\} - m_{yz} \left\{ xq f - f \right\} \]

\[ + m_y \left\{ xq f - f \right\} + m_z \left\{ xq f - f \right\} - m_{yz} \left\{ xq f - f \right\} \]

\[ = -\phi(p) - \phi(q) - \phi(x) + \phi(xp) + \phi(xq) + \phi(pq), \]
therefore \( \varphi \) is a multiplicative Deeba function. We now get

\[
|\varphi(x) - f(x)| = |m_y \{ f - f \} + m_z \{ f - f \} - m_{yz} \{ f - f \} - f(x)| \\
\leq \|m\| |f(xy) - f(y) + f(xz) - f(z)| \\
- |f(xy) + f(yz) - f(x)| \\
\leq |f(xyz) + f(x) + f(y) + f(z)| \\
- |f(xy) - f(xz) - f(yz)| \\
\leq \delta,
\]

for all \( x \in G \). This completes the proof of theorem.

References